MATH 2060

22. Show that if $f_n(x) := x + 1/n$ and $f(x) := x$ for $x \in \mathbb{R}$, then (f_n) converges uniformly on $\mathbb{R}$ to f, but the sequence (f_n^2) does not converge uniformly on $\mathbb{R}$. (Thus the product of uniformly convergent sequences of functions may not converge uniformly.)

Pf: $\forall n \in \mathbb{N}$,

$$|f_n(x) - f(x)| = |x + 1/n - x| = \frac{1}{n}, \quad \forall x \in \mathbb{R}$$

$$\Rightarrow \quad \|f_n - f\|_{\mathbb{R}} = \frac{1}{n} \to 0 \quad \text{as } n \to \infty$$

Hence (f_n) converges uniformly on $\mathbb{R}$ to f.

OTOH, $\forall n \in \mathbb{N}$, $\forall x \in \mathbb{R}$,

$$|f_n^2(x) - f^2(x)| = |(x + 1/n)^2 - x^2|$$
$$= \left|2\frac{x}{n} + \frac{1}{n^2}\right|$$

In particular, if $x = n$, then

$$|f_n^2(n) - f^2(n)| = 2 + \frac{1}{n^2} \geq 2$$

So $\|f_n^2 - f^2\|_{\mathbb{R}} \geq 2$

$$\Rightarrow \quad \|f_n^2 - f^2\|_{\mathbb{R}} \not\to 0 \quad \text{as } n \to \infty$$

Hence (f_n^2) does not converge uniformly on $\mathbb{R}$ to f^2

$\Rightarrow$ (f_n^2) does not converge uniformly on $\mathbb{R}$ since

$$\lim_{n\to\infty} f_n^2(x) = f^2(x) \quad \forall x \in \mathbb{R}$$

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5. Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $\mathbb{R}$ and let $f_n(x) := f(x + 1/n)$ for $x \in \mathbb{R}$. Show that (f_n) converges uniformly on $\mathbb{R}$ to f.

Pf: Let $\varepsilon > 0$.

Since f is uniformly cts on $\mathbb{R}$,

$\exists\, \delta > 0$ s.t. if $x, y \in \mathbb{R}$ and $|x-y| < \delta$,

then $|f(x) - f(y)| < \varepsilon$

Note $|f_n(x) - f(x)| = |f(x+\frac{1}{n}) - f(x)|$

Choose $N \in \mathbb{N}$ s.t. $1/N < \delta$.

Now, $\forall\, n \geq N$, $\forall\, x \in \mathbb{R}$, we have

$$|(x + \tfrac{1}{n}) - x| = \tfrac{1}{n} \leq \tfrac{1}{N} < \delta$$

and so $|f_n(x) - f(x)| = |f(x+\frac{1}{n}) - f(x)| < \varepsilon$

Therefore $f_n \rightrightarrows f$ on $\mathbb{R}$ //

7. Suppose the sequence (f_n) converges uniformly to f on the set A, and suppose that each f_n is bounded on A. (That is, for each n there is a constant M_n such that $|f_n(x)| \leq M_n$ for all $x \in A$.) Show that the function f is bounded on A.

Pf: Idea: $\forall n \in \mathbb{N}$, $\forall x \in A$,

$$\begin{aligned} |f(x)| &= |f(x) - f_n(x) + f_n(x)| \\ &\leq \underbrace{|f(x) - f_n(x)|}_{\text{small for large } n} + \underbrace{M_n}_{\text{need fixed } n} \end{aligned}$$

Take $\varepsilon_0 = 1$.

Since $f_n \rightrightarrows f$ on A,

$\exists N \in \mathbb{N}$ s.t. if $n \geq N$, then

$$|f_n(x) - f(x)| < \varepsilon_0 \qquad \forall x \in A$$

In particular, $|f_N(x) - f(x)| < \varepsilon_0 = 1 \qquad \forall x \in A$

So, $\forall x \in A$,

$$\begin{aligned} |f(x)| &\leq |f_N(x) - f(x)| + |f_N(x)| \\ &< 1 + M_N \\ &=: M \end{aligned}$$

Hence f is bounded on A ✓

15. Let $g_n(x) := nx(1-x)^n$ for $x \in [0,1]$, $n \in \mathbb{N}$. Discuss the convergence of (g_n) and $(\int_0^1 g_n dx)$.

(This almost the same seq of fcns considered in last tutorial)

Ans: · Note $\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} nx(1-x)^n = 0 \quad \forall x \in [0,1]$

So $g_n \to g := 0$ pointwise on $[0,1]$.

· OTOH, for $n \geq 2$,

$$g_n'(x) = n(1-x)^n + nx(-n)(1-x)^{n-1} = n(1-x)^{n-1}(1-(n+1)x)$$

Thus $g_n'(x) = 0 \iff x = 1$ or $x = \frac{1}{n+1}$

So $|g_n(x) - g(x)| = g_n(x)$ attains it max on $[0,1]$

at $x = 0$, $x = 1$ or $x = \frac{1}{n+1}$

Compare $g_n(0) = 0$ $\quad g_n(1) = 0$ $\quad g_n(\frac{1}{n+1}) = n(\frac{1}{n+1})(1-\frac{1}{n+1})^n$

$$= \left(\frac{n}{n+1}\right)^{n+1}$$

$$\Rightarrow \|g_n - g\|_{[0,1]} = g_n\left(\frac{n}{n+1}\right) = \left(1 - \frac{1}{n+1}\right)^{n+1} \to \frac{1}{e} \neq 0$$

Hence (g_n) does not converge uniformly on $[0,1]$.

· $$\int_0^1 g_n(x)\,dx = \int_0^1 nx(1-x)^n\,dx$$

$$= n\int_1^0 \varphi(y)(1-\varphi(y))^n \cdot \varphi'(y)\,dy$$

Let $y = 1-x$

or $\varphi : [0,1] \to \mathbb{R}$

$\varphi(y) := 1-y$, C^1

$$= n\int_1^0 (1-y)\,y^n \cdot (-1)\,dy$$

$$= n\int_0^1 (y^n - y^{n+1})\,dy$$

$$= n\left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{n}{(n+1)(n+2)} \to 0$$

which is the same as $\int_0^1 g(x)\,dx = 0$ //

Ex Let $f_n \in C[a,b]$ converge pointwisely to f on $[a,b]$. Suppose that $f_n \rightrightarrows f$ on (a,b). Show that $f \in C[a,b]$ and $f_n \rightrightarrows f$ on $[a,b]$.

Pf: $\begin{cases} f_n \longrightarrow f & \text{on } [a,b] \quad ① \\ f_n \rightrightarrows f & \text{on } (a,b) \quad ② \end{cases} \Rightarrow f_n \rightrightarrows f$ on $[a,b]$

Let $\varepsilon > 0$.

① $\Rightarrow \exists\, N_1, N_2 \in \mathbb{N}$ s.t. $|f_n(a) - f(a)| < \varepsilon \quad \forall\, n \geq N_1$

$|f_n(b) - f(b)| < \varepsilon \quad \forall\, n \geq N_2$.

② $\Rightarrow \exists\, N \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \varepsilon \quad \forall\, x \in (a,b), \forall\, n \geq N$.

Take $N' := \max\{N_1, N_2, N\}$.

If $n \geq N'$, then $|f_n(x) - f(x)| < \varepsilon \quad \forall\, x \in [a,b]$

So $f_n \rightrightarrows f$ on $[a,b]$.

Recall: (Interchange of Limit and Continuity)

Let (f_n) be a seq of cts fcn on $A \subseteq \mathbb{R}$

If $f_n \rightrightarrows f$ on A, then f is cts on A.

Since each f_n is cts on $[a,b]$ and $f_n \rightrightarrows f$ on $[0,1]$,

f is cts on $[a,b]$

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Ex Let f be a continuously differentiable function defined on (a,b) (i.e. f' is continuous). Consider the sequence (f_n):

$$f_n(x) = n\left(f\left(x+\frac{1}{n}\right) - f(x)\right)$$

Show that f_n converges uniformly to f' in any closed subinterval $[c,d]$ of (a,b).

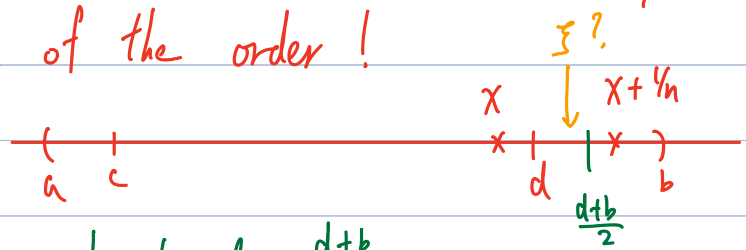
Pf: Idea: By MVT, $f_n(x) = \dfrac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n} - 0} = f'(\xi)$

for some $\xi = \xi(n,x) \in (x, x+\frac{1}{n})$

So $|f_n(x) - f'(x)| = |\underbrace{f'(\xi) - f'(x)}|$

small since f' is cts, hence uniformly cts on any $[c,d] \subseteq (a,b)$

Becareful of the order!



Let $\varepsilon > 0$. Let $\ell = \frac{d+b}{2}$.

Since f' is cts and hence uniformly cts on $[c,\ell]$,

$\exists\ \delta > 0$ s.t. if $u, v \in [c,\ell]$ and $|u-v| < \delta$, (#)

then $|f'(u) - f'(v)| < \varepsilon$

Take $\delta' = \min\{\ell - d, \delta\} > 0$.

Choose $N \in \mathbb{N}$ s.t. $1/N < \delta'$ (so $x \in [c,d] \Rightarrow x + \frac{1}{N} \in [c,\ell]$)

$\forall x \in [c,d]$, $\forall n \geq N$, by MVT,

$\exists\ \xi = \xi(n,x) \in (x, x+\frac{1}{n}) \subseteq [c,\ell]$ s.t.

$$f'(\xi) = n\left(f(x+\tfrac{1}{n}) - f(x)\right) = f_n(x)$$

Since $x, \xi \in [c,\ell]$ and $|x - \xi| < \frac{1}{n} \leq \frac{1}{N} < \delta'$, we have

$$|f_n(x) - f'(x)| = |f'(\xi) - f'(x)| < \varepsilon \qquad \text{by (\#)}$$

Therefore $f_n \rightrightarrows f'$ on $[c,d]$ //